Self-dual Smooth Approximations of Convex Functions via the Proximal Average

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Abstract

The proximal average of two convex functions has proven to be a useful tool in convex analysis. In this note, we express Goebel's self-dual smoothing operator in terms of the proximal average, which allows us to give a simple proof of self duality. We also provide a novel self-dual smoothing operator. Both operators are illustrated by smoothing the norm.

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1 Introduction

Let X be the standard Euclidean space $\mathbb{R}^n$, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. It will be convenient to set

$$\mathfrak{q} = \tfrac{1}{2}\|\cdot\|^2. \tag{1}$$

Now let $f\colon X \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper. Since many convex functions are nonsmooth, it is natural to ask: How can one approximate f with a smooth function?

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The most famous and very useful answer to this question is provided by the *Moreau envelope* [13, 15], which, for $\lambda > 0$, is defined by [1]

$$e_\lambda f = f \,\Box\, \lambda^{-1}\mathfrak{q}. \tag{2}$$

It is well known that $e_\lambda f$ is smooth and that $\lim_{\lambda\to 0^+} e_\lambda f = f$ point-wise; see, e.g., [15, Theorem 1.25 and Theorem 2.26].

Let us consider the norm, which is nonsmooth at the origin.

Example 1.1 (Moreau envelope of the norm) Let $\lambda \in \,]0,1[$, set $f = \|\cdot\|$, and denote the closed unit ball by C. Then, for x and x^* in X, we have [2]

$$e_\lambda f(x) = \begin{cases} \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \lambda; \\[2ex] \|x\| - \dfrac{\lambda}{2}, & \text{if } \|x\| > \lambda, \end{cases} \tag{3}$$

$\big(e_\lambda f\big)^* = \iota_C + \lambda\mathfrak{q}$, and $e_\lambda (f^*)(x^*) = (2\lambda)^{-1}\cdot\big(\max\{0, \|x^*\|-1\}\big)^2$. Consequently, $\big(e_\lambda f\big)^* \neq e_\lambda (f^*)$.

Proof. Either a straight-forward computation or [15, Example 11.26(a)] yields

$$f^* = \iota_C. \tag{4}$$

Next, if $y \in X$, then

$$e_{1/\lambda}\, \iota_C(y) = \inf_{c\in C} \lambda\mathfrak{q}(y-c) \tag{5}$$

$$= \frac{\lambda}{2} d_C^2(y) \tag{6}$$

$$= \frac{\lambda}{2}\cdot \begin{cases} \big(\|y\|-1\big)^2, & \text{if } \|y\| > 1; \\[2ex] 0, & \text{if } \|y\| \le 1, \end{cases} \tag{7}$$

and thus

$$e_{1/\lambda}\, \iota_C\big(x/\lambda\big) = \frac{\lambda}{2}\cdot \begin{cases} \big(\|x/\lambda\|-1\big)^2, & \text{if } \|x\| > \lambda; \\[2ex] 0, & \text{if } \|x\| \le \lambda. \end{cases} \tag{8}$$

[1] The symbol "$\Box$" denotes *infimal convolution*: $(f_1 \,\Box\, f_2)(x) = \inf_y \big(f_1(y) + f_2(x-y)\big)$.

[2] Here $\iota_C(x) = 0$, if $x \in C$; $\iota_C(x) = +\infty$, if $x \notin C$ is the *indicator function*, $f^*(x^*) = \sup_{x\in X} \big(\langle x, x^*\rangle - f(x)\big)$ is the *Fenchel conjugate* of f, and $d_C = \|\cdot\| \,\Box\, \iota_C$ is the *distance function*.

By [15, Example 11.26(b) on page 495], we obtain

$$
\begin{aligned}
e_\lambda f(x) &= \frac{1}{\lambda}\mathfrak{q}(x) - e_{1/\lambda} f^*\big(x/\lambda\big) && (9)\\
&= \frac{1}{2\lambda}\|x\|^2 - \frac{\lambda}{2}\cdot\begin{cases}\dfrac{\|x\|^2}{\lambda^2} - \dfrac{2\|x\|}{\lambda} + 1, & \text{if } \|x\| > \lambda;\\ 0, & \text{if } \|x\| \le \lambda\end{cases} && (10)\\
&= \begin{cases}\|x\| - \dfrac{\lambda}{2}, & \text{if } \|x\| > \lambda;\\ \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \lambda\end{cases} && (11)
\end{aligned}
$$

and $\big(e_\lambda f\big)^* = f^* + \lambda\mathfrak{q} = \iota_C + \lambda\mathfrak{q}$. Alternatively, one may use [6, Example 2.16], which provides the proximal mapping of f, and then use the proximal mapping calculus to obtain these results. ■

While the Moreau envelope has many desirable properties, we see from Example 1.1 that the smooth approximation $e_\lambda f$ is not *self-dual* in the sense that

$$\big(e_\lambda f\big)^* \neq e_\lambda\,(f^*). \tag{12}$$

It is perhaps surprising that self-dual smoothing operators even exist. The first example appears in [9]. Specifically, Goebel defined

$$G_\lambda f = (1-\lambda^2)e_\lambda f + \lambda\mathfrak{q} \tag{13}$$

and proved that

$$\big(G_\lambda f\big)^* = G_\lambda\,(f^*), \tag{14}$$

i.e., *Fenchel conjugation and Goebel smoothing commute!* For applications of his smoothing operator, see [9].

The purpose of this note is two-fold. First, we present a different representation of the Goebel smoothing operator which allows us to prove self-duality using the Fenchel conjugation formula for the proximal average. Secondly, the proximal average is also utilized to obtain a novel smoothing operator. Both smoothing operators are computed explicitly for the norm. The formulas derived show that the new smoothing operator is distinct from the one provided by Goebel.

For f_1 and f_2, two functions from X to $\left]-\infty,+\infty\right]$ that are convex, lower semicontinuous and proper, and for two strictly positive convex coefficients ($\lambda_1 + \lambda_2 = 1$), the *proximal average* is defined by

$$\operatorname{pav}(f_1, f_2\,; \lambda_1, \lambda_2) = \big(\lambda_1(f_1+\mathfrak{q})^* + \lambda_2(f_2+\mathfrak{q})^*\big)^* - \mathfrak{q}. \tag{15}$$

See [1, 2, 3, 4, 5, 9, 10] for further information and applications of the proximal average. The key property is the *Fenchel conjugation formula*

$$\operatorname{pav}(f_1, f_2\,; \lambda_1, \lambda_2)^* = \operatorname{pav}(f_1^*, f_2^*\,; \lambda_1, \lambda_2)\,; \tag{16}$$

see [4, Theorem 6.1], [2, Theorem 4.3], or [1, Theorem 5.1].

We use standard convex analysis calculus and notation as, e.g., in [14, 15, 18]. In Section 2, we consider Goebel's smoothing operator from the proximal-average view point. The new smoothing operator is presented in Section 3.

2 The Goebel smoothing operator

Definition 2.1 (Goebel smoothing operator) *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex, lower semicontinuous and proper, and let $\lambda \in \left]0,1\right[$. Then the* Goebel smoothing operator [9] *is defined by*

$$G_\lambda f = (1-\lambda^2)e_\lambda f + \lambda\mathfrak{q}. \tag{17}$$

Note that (17) and standard properties of the Moreau envelope imply that point-wise

$$\lim_{\lambda\to 0^+} G_\lambda f = f \tag{18}$$

and that each $G_\lambda f$ is smooth.

Our first main result provides two alternative descriptions of the Goebel smoothing operator. The first description, item (i) in Theorem 2.2, shows a pleasing reformulation in terms of the proximal average. The second description, item (ii) in Theorem 2.2 is less appealing but has the advantage of providing a simple proof of the *self-duality* (iii) observed by Goebel.

Theorem 2.2 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex, lower semicontinuous and proper, and let $\lambda \in \left]0,1\right[$. Then the following hold*[3].

(i) $G_\lambda f = (1+\lambda)\operatorname{pav}(f,0\,;1-\lambda,\lambda) + \lambda\mathfrak{q}$.

(ii) $G_\lambda f = (1+\lambda)^2\operatorname{pav}\Big(f,\mathfrak{q}\,;\frac{1-\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}\Big)\circ(1+\lambda)^{-1}\operatorname{Id}$.

(iii) **(Goebel)** $\big(G_\lambda f\big)^* = G_\lambda(f^*)$.

[3] Here $\operatorname{Id}\colon X\to X\colon x\mapsto x$ is the *identity operator*.

Proof. Let $x \in X$. Then, using (15) and standard convex calculus, we obtain

$$\Big((1+\lambda)^2 \operatorname{pav}\Big(f, \mathfrak{q}\,; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\Big) \circ (1+\lambda)^{-1}\operatorname{Id}\Big)(x) \tag{19}$$

$$= (1+\lambda)^2 \bigg(\Big(\frac{1-\lambda}{1+\lambda}(f+\mathfrak{q})^* + \frac{2\lambda}{1+\lambda}(\mathfrak{q}+\mathfrak{q})^*\Big)^* - \mathfrak{q}\bigg)\Big(\frac{x}{1+\lambda}\Big) \tag{20}$$

$$= (1+\lambda)^2 \Big(\frac{1-\lambda}{1+\lambda}(f+\mathfrak{q})^* + \frac{\lambda}{1+\lambda}\mathfrak{q}\Big)^* \Big(\frac{x}{1+\lambda}\Big) - \mathfrak{q}(x) \tag{21}$$

$$= (1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\mathfrak{q}\Big)^*(x) - \mathfrak{q}(x) \tag{22}$$

$$= (1+\lambda)\bigg(\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\big(0+\mathfrak{q}\big)^*\Big)^* - \mathfrak{q}\bigg)(x) + \lambda\mathfrak{q}(x) \tag{23}$$

$$= \Big((1+\lambda)\operatorname{pav}(f, 0\,; 1-\lambda, \lambda) + \lambda\mathfrak{q}\Big)(x). \tag{24}$$

We have verified that (22) as well as the right sides of (i) and (ii) coincide. Starting from (22) and again applying standard convex caluclus, we see that

$$(1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\mathfrak{q}\Big)^*(x) - \mathfrak{q}(x) \tag{25}$$

$$= (1+\lambda)\Big(\Big((1-\lambda)\big(f+\mathfrak{q}\big)^*\Big)^* \,\square\, \big(\lambda\mathfrak{q}\big)^*\Big)(x) - \mathfrak{q}(x) \tag{26}$$

$$= (1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)\Big(\frac{\cdot}{1-\lambda}\Big) \,\square\, \frac{1}{\lambda}\mathfrak{q}\Big)(x) - \mathfrak{q}(x) \tag{27}$$

$$= (1+\lambda)\inf_y \Big((1-\lambda)\big(f+\mathfrak{q}\big)\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda}\mathfrak{q}(x-y)\Big) - \mathfrak{q}(x) \tag{28}$$

$$= (1+\lambda)\inf_y \Big((1-\lambda)f\Big(\frac{y}{1-\lambda}\Big) + (1-\lambda)\mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda}\mathfrak{q}(x-y) - \frac{1}{1+\lambda}\mathfrak{q}(x)\Big) \tag{29}$$

$$= (1-\lambda^2)\inf_y \Big(f\Big(\frac{y}{1-\lambda}\Big) + \mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda^2}\mathfrak{q}(x)\Big). \tag{30}$$

Simple algebra shows that for every $y \in X$,

$$\mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda^2}\mathfrak{q}(x) = \frac{1}{\lambda}\mathfrak{q}\Big(x - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{1-\lambda^2}\mathfrak{q}(x). \tag{31}$$

Therefore,

$$(1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\mathfrak{q}\Big)^*(x) - \mathfrak{q}(x) \tag{32}$$

$$= (1-\lambda^2)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda^2}\mathfrak{q}(x)\right) \tag{33}$$

$$= (1-\lambda^2)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \frac{1}{\lambda}\mathfrak{q}\Big(x - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{1-\lambda^2}\mathfrak{q}(x)\right) \tag{34}$$

$$= (1-\lambda^2)\inf_z \left(f(z) + \frac{1}{\lambda}\mathfrak{q}(x-z) + \frac{\lambda}{1-\lambda^2}q(x)\right) \tag{35}$$

$$= \big((1-\lambda^2)e_\lambda f + \lambda\mathfrak{q}\big)(x) \tag{36}$$

$$= G_\lambda f(x), \tag{37}$$

which completes the proof of (i) and (ii).

(iii): In view of the conjugate formula $(\beta^2 h \circ (\beta^{-1}\,\mathrm{Id}))^* = \beta^2 h^* \circ (\beta^{-1}\,\mathrm{Id})$, (ii), and (16), we obtain

$$\big(G_\lambda f\big)^* = \left((1+\lambda)^2 \operatorname{pav}\left(f, \mathfrak{q}\,; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right) \circ (1+\lambda)^{-1}\,\mathrm{Id}\right)^* \tag{38}$$

$$= (1+\lambda)^2 \left(\operatorname{pav}\left(f, \mathfrak{q}\,; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right)\right)^* \circ (1+\lambda)^{-1}\,\mathrm{Id} \tag{39}$$

$$= (1+\lambda)^2 \operatorname{pav}\left(f^*, \mathfrak{q}^*\,; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right) \circ (1+\lambda)^{-1}\,\mathrm{Id} \tag{40}$$

$$= (1+\lambda)^2 \operatorname{pav}\left(f^*, \mathfrak{q}\,; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right) \circ (1+\lambda)^{-1}\,\mathrm{Id} \tag{41}$$

$$= G_\lambda\,(f^*). \tag{42}$$

The proof is complete. ■

Remark 2.3 Theorem 2.2(i)&(ii) gives two representations of the Goebel smoothing operator in terms of the proximal average. Goebel [8] discovered a converse formula, which we state next without proof:

$$\operatorname{pav}\,(f, \mathfrak{q}\,; \lambda, 1-\lambda) = \frac{(2-\lambda)^2}{4} G_{\lambda/(2-\lambda)}\, f \circ \Big(\frac{2}{2-\lambda}\,\mathrm{Id}\Big). \tag{43}$$

Example 2.4 Let $\lambda \in\,]0,1[$ and set $f = \|\cdot\|$. Then, for every $x \in X$,

$$G_\lambda f(x) = \begin{cases} \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \lambda; \\ \dfrac{\lambda\|x\|^2}{2} + (1-\lambda^2)\|x\| - \dfrac{\lambda(1-\lambda^2)}{2}, & \text{if } \|x\| > \lambda. \end{cases} \tag{44}$$

Proof. Combine (17) and (3). ■

3 A new smoothing operator

We now provide a novel smoothing operator that has a very simple expression in terms of the proximal average.

Definition 3.1 (new smoothing operator) *Let $f\colon X \to\,]-\infty,+\infty]$ be convex, lower semicontinuous and proper, and let $\lambda \in\,]0,1[$. Then the $S_\lambda f$ is defined by*

$$S_\lambda f = \operatorname{pav}(f, \mathsf{q}\,; 1-\lambda, \lambda)\,. \tag{45}$$

Theorem 3.2 *Let $f\colon X \to\,]-\infty,+\infty]$ be convex, lower semicontinuous and proper, and let $\lambda \in\,]0,1[$. Set $\mu = \lambda/(2-\lambda)$. Then the following hold.*

(i) $S_\lambda f = (1-\lambda)e_\mu f \circ \left(\frac{2}{2-\lambda}\operatorname{Id}\right) + \mu\mathsf{q}$.

(ii) $\big(S_\lambda f\big)^* = S_\lambda(f^*)$.

Proof. (i): Let $x \in X$. Then, using (45), (15) and standard convex calculus, we obtain

$$\big(S_\lambda f\big)(x) = \big((1-\lambda)(f+\mathsf{q})^* + \lambda(\mathsf{q}+\mathsf{q})^*\big)^*(x) - \mathsf{q}(x) \tag{46}$$

$$= \big((1-\lambda)(f+\mathsf{q})^* + \tfrac{\lambda}{2}\mathsf{q}\big)^*(x) - \mathsf{q}(x) \tag{47}$$

$$= \left((1-\lambda)(f+\mathsf{q})\Big(\frac{\cdot}{1-\lambda}\Big) \,\square\, \frac{2}{\lambda}\mathsf{q}\right)(x) - \mathsf{q}(x) \tag{48}$$

$$= \inf_y \left((1-\lambda)f\Big(\frac{y}{1-\lambda}\Big) + (1-\lambda)\mathsf{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda}\mathsf{q}(x-y) - \mathsf{q}(x)\right) \tag{49}$$

$$= (1-\lambda)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \mathsf{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda(1-\lambda)}\mathsf{q}(x-y) - \frac{1}{1-\lambda}\mathsf{q}(x)\right). \tag{50}$$

Simple algebra shows that for every $y \in X$,

$$\mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda}\mathfrak{q}(x) = \frac{2-\lambda}{\lambda}q\Big(\frac{2x}{2-\lambda} - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{(1-\lambda)(2-\lambda)}\mathfrak{q}(x). \tag{51}$$

Therefore,

$$\big(S_\lambda\, f\big)(x) = (1-\lambda)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \frac{2-\lambda}{\lambda}q\Big(\frac{2x}{2-\lambda} - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{(1-\lambda)(2-\lambda)}\mathfrak{q}(x)\right) \tag{52}$$

$$= (1-\lambda)\inf_z \left(f(z) + \frac{2-\lambda}{\lambda}q\Big(\frac{2x}{2-\lambda} - z\Big)\right) + \frac{\lambda}{2-\lambda}\mathfrak{q}(x) \tag{53}$$

$$= (1-\lambda)\Big(f\,\square\,\frac{1}{\mu}\mathfrak{q}\Big)\Big(\frac{2x}{2-\lambda}\Big) + \mu\mathfrak{q}(x), \tag{54}$$

as claimed.

(ii): Using (45) and (16), we get

$$\big(S_\lambda\, f\big)^* = \big(\operatorname{pav}(f,\mathfrak{q}\,;1-\lambda,\lambda)\big)^* = \operatorname{pav}(f^*,\mathfrak{q}^*\,;1-\lambda,\lambda) = \operatorname{pav}(f^*,\mathfrak{q}\,;1-\lambda,\lambda) = S_\lambda\,(f^*). \tag{55}$$

The proof is complete. ■

Note that Theorem 3.2(i) and standard properties of the Moreau envelope imply that point-wise

$$\lim_{\lambda\to 0^+} S_\lambda\, f = f \tag{56}$$

and that each $S_\lambda\, f$ is smooth.

Example 3.3 Let $\lambda \in\,]0,1[$ and set $f = \|\cdot\|$. Then, for every $x \in X$,

$$S_\lambda\, f(x) = \begin{cases} \dfrac{(2-\lambda)\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \dfrac{\lambda}{2};\\[2mm] \dfrac{\lambda\|x\|^2}{2(2-\lambda)} + \dfrac{2(1-\lambda)}{2-\lambda}\|x\| - \dfrac{\lambda(1-\lambda)}{2(2-\lambda)}, & \text{if } \|x\| > \dfrac{\lambda}{2}. \end{cases} \tag{57}$$

Proof. Combine (3) and Theorem 3.2(i). ■

Remark 3.4 Let $f = \|\cdot\|$. The explicit formulas provided in Example 2.4 and Example 3.3 imply that $G_\alpha\, f \neq S_\beta\, f$, for *all* α and β in $]0,1[$. Thus, the smoothing operator defined by (45) is indeed new and different from Goebel's smoothing operator.

Remark 3.5 Given a more complicated function f, the explicit computation of the smoothing operators $G_\lambda\, f$ and $S_\lambda\, f$ may not be so easy. However, computational convex analysis provides tools [11, 12] to compute the Moreau envelope numerically which — due to the Moreau envelope formulations (17) and Theorem 3.2(i) — makes it possible to compute the smoothing operators $G_\lambda\, f$ and $S_\lambda\, f$ numerically.

Finally, other approaches to smooth approximation are: Ghomi's integral convolution method [7], Seeger's ball rolling technique [16], and Teboulle's entropic proximal mappings [17].

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